Discrete Energies of a Weakly Outcoupled Atom Laser Beam Outside the Bose–Einstein Condensate Region

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Abstract

We consider the possibility of a discrete set of energies of a weakly outcoupled atom laser beam to the homogeneous Schrödinger equation with anisotropic harmonic trap in Cartesian coordinates outside the Bose–Einstein condensate region. This treatment is used because working in the cylindrical coordinates is not really possible, even though we implement the cigar-shaped trap case. The Schrödinger equation appears to replace a set of two-coupled Gross–Pitaevskii equations by enabling the weak-coupling assumption. This atom laser can be produced in a simple way that only involves extracting the atoms in a condensate from by using the radio frequency field. We initially present the relation between condensates as sources and atom laser as an output by exploring the previous work of Riou et al. in the case of theoretical work for the propagation of atom laser beams. We also show that even though the discrete energies are obtained by means of an approaching harmonic oscillator, degeneracy is only available in two states because of the anisotropic external potential.

Keywords: atom laser, Bose–Einstein condensation, harmonic oscillator

1. Introduction

It is a well-known fact that Bose–Einstein condensation (BEC) is considered a worthwhile theory for studying cold atoms at very low temperatures, in the $T \approx 0$ K region [1-4]. This theory was proposed a long time ago by Bose and Einstein [5-7], and it has been verified by several experiments using alkali atoms [8-10]. To attain BEC, atoms confined in a magnetic trap are cooled below the critical temperature until the wave packets overlap each other; a successful result can be examined by observing the sharp peak in velocity distribution [5]. For mathematical criteria, see, for example, [11-12]. This phenomenon has led physicists to use Bose–Einstein condensates to produce the physical properties. One of the famous uses of the condensates is atom laser production [13].

Atom laser is actually a type of matter wave that has similar properties as ordinary laser, such as coherency. However, the significant difference between atom laser and ordinary laser lies in the propagation. The
propagation of ordinary laser is not influenced by gravity, as the photon is massless; otherwise, because the atom laser is massive, the de Broglie wavelength decreases monotonically, due to the acceleration of gravity [14-18]. For some experiments related to atom laser, see [19-21]. It typically requires some type of apparatus to reduce the effect of gravity when the atom laser leaves confinement [18]. This confinement is actually produced by a magnetic field that is usually represented by a three-dimensional anisotropic harmonic oscillator. However, the atom laser is of major interest for applications; for example, see [21-25].

Generally, a weakly outcoupled atom laser can be created by pulsing a radio frequency as an outcoupler to a set of condensates in confinement, until two or more atoms in the condensate are released from the trap with a certain kinetic energy. In quantum language, the atoms are simply flipped from the magnetically trapped state to an untrapped state. Consequently, the atom laser is outcoupled from a condensate as a source, and because the atom is massive, the atom laser has a significant mass. Theoretically, to discuss the propagation of atom lasers, we must include the propagation of condensate as well. As such, the related Gross–Pitaevskii equations (GPEs) include both the condensate and the atom laser and their coupling term. In the next section, we enable an available assumption by reviewing the previous work of Riou et al. [26] in order to simplify the case.

Because most previous authors were concerned with the density profile of the atom laser beam wave function [13, 15, 19, 20, 26], in this paper, we emphasize the derivation of a set of discrete energies of atom laser beam based on the propagation of atom laser outside the BEC region, which can be used to examine the quality of the beam. The rest of this paper is organized as follows. We briefly review the previous work of Riou et al. [26] in reducing GPE into the Schrödinger equation in Sec. 2. In Sec. 3, we construct the discrete energies using the quantum oscillator approach. We close our discussion in Sec. 4 with conclusions based on our final results.

2. Methods

In this section, we are concerned with the available equation of an atom laser beam, which is derived from a set of two coupled GPEs. This set can be written as [26]

\[
\frac{i\hbar}{\partial t} \psi_i = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_i(\vec{r}) + \sum_{k\neq l} g_{kl} |\psi_j|^2 \right) \psi_i + W_j(\vec{r},t)\psi_j, \tag{1}
\]

where \(i = \{\ell, s\}\), \(V_i\) denotes anisotropic external potential, \(g_{kl}\) is an interaction strength between two particles in a condensate, and \(W_j\) represents a coupling function between the wave function of condensate \(\psi_i\) and atom laser beam wave function \(\psi_s\).

Because of the nonlinearity attached in \(g_{kl}\), the available analytical atom laser beam solution is difficult. However, by proposing a certain condition, the above equation (1) can be simplified. In their previous paper [26], assuming that atom laser propagation is weakly outcoupled from the condensate, Riou et al. stated that the reduction of Eq. (1) will produce an inhomogeneous Schrödinger equation representing atom laser beam dynamics if the weak coupling condition is acceptable. According to the authors, reducing Eq. (1) by applying the condition will produce a GPE representing the dynamics of the condensate [26]

\[
i\hbar \frac{\partial \psi_s}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_s + g_{s,\ell} |\psi_\ell|^2 \right) \psi_s \tag{2}
\]

and the Schrödinger equation of the atom laser beam

\[
i\hbar \frac{\partial \psi_\ell}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_\ell + g_{s,\ell} |\psi_s|^2 \right) \psi_\ell + \rho, \tag{3}
\]

where \(\rho(\vec{r},t)\) is considered a source term containing the coupling term \(W_i\). By accepting the above assumption, we have the benefit of finding the solution to the atom laser beam, even though there is still a coupling term between \(\psi_\ell\) and \(\psi_s\). It has also been verified that Eq. (2) is the ordinary GPE whose solution has been explored in some papers; for example, see [2, 4].

Propagation of an atom laser beam typically experiences two stages. For \(\rho \neq 0\), the propagation occurs inside the BEC region with the external potential with the different sign of our convention with [26]

\[
V_i(\vec{r}) = -\mu + \frac{1}{2} m [\omega^2_1 (x^2 + y^2) + \omega^2_2 z^2] + \frac{1}{2} m \omega^2_3 \sigma^2 \tag{4}
\]

and outside for \(\rho = 0\)

\[
V_i(\vec{r}) = -\mu + \frac{1}{2} m \omega^2_1 \sigma^2 + \frac{1}{2} m \omega^2_2 \sigma^2 + \frac{1}{2} m \omega^2_3 \sigma^2 \tag{5}
\]

where \(\mu\) is the chemical potential, and the sags due to gravity potential, \(-mg\), are stated by \(\sigma = g / \omega^2_r\) and \(\sigma = g / \omega^2_z\). In the case of a cigar-shaped or disk-shaped trap, the relationship among \(\omega\), \(\omega_1\), and \(\omega_2\) can usually be given by \(\omega = (\omega_1^2 + \omega_2^2)^{1/2}\) [5]. For the inside region, the appearance of \(\rho\) leads to an inhomogeneous Schrödinger equation whose exact solution can only be found if \(\rho\) is determined [27].

Here, we are interested in solving the dynamics of the atom laser beam, focusing on seeking available energy states outside the BEC region (\(\rho = 0\)) by solving a homogeneous Schrödinger equation in Eq. (3). Observing the external potential in Eq. (5), we suspect that the system has discrete energies if harmonic oscillator in quantum mechanics is considered. While we use a cigar-shaped trap (\(\omega_1 / \omega_2 \ll 1\)) in this paper,
the cylindrical coordinates cannot be used, as separation of variables is not available.

3. Results and Discussion

To start the discussions, we reconsider Eq. (3) for the case \( \rho = 0 \)

\[
\psi_t = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_t + g_{sf} |\psi_t|^2 \right] \psi_t.
\]

Because it is convenient to work in a dimensionless form, we make the following transformations:

\[
(x, y, z) \rightarrow (x, y, z)/a_0,
\]

\[
t \rightarrow \frac{t}{2(\alpha \sigma \sqrt{\lambda})},
\]

\[
\psi_t \rightarrow \psi_t \sqrt{\frac{2g_{sf} \sqrt{\lambda}}{\hbar a_0}},
\]

where \( a_0 = \sqrt{\hbar / m \sigma \alpha \sqrt{\lambda}} \) and \( \lambda = \alpha^2 / a_0^2 \). Substituting Eqs. (7) and (8) into Eq. (6), we attain the dimensionless form

\[
\frac{1}{2} \frac{d^2 \psi_t}{dx^2} - \lambda^{-1} x^2 = -E_x,
\]

\[
\frac{1}{2} \frac{d^2 \psi_t}{dy^2} - \lambda^{-1} y^2 = -E_y,
\]

\[
\frac{1}{2} \frac{d^2 \psi_t}{dz^2} - \alpha |\psi_t|^2 = -E_z,
\]

where \( E_x, E_y, \) and \( E_z \) are constants and obey

\[
E = E_x + E_y + E_z.
\]

If we ignore the boundary condition, the solutions of Eqs. (11) and (12) will not achieve discrete energies, and the appropriate eigenfunctions should be the parabolic cylinder function. Because we are only interested in discrete energies, we apply the normalization condition over the space.

Note that each of equations (12)–(14) has a similar form as the quantum oscillator; thus, it is beneficial to adjust all of the equations into an oscillator-like equation. If we transform \( \epsilon = (\lambda^{-1})^{1/4} x \) in Eq. (11), by pursuing the standard method in the quantum mechanics textbook, we obtain discrete energy:

\[
(E_x)_m = \sqrt{\lambda^{-1}} (2m + 1).
\]

Then, if we use a little trick by changing Eq. (12) with

\[
\frac{d^2 Y}{d\xi^2} - \lambda^{-1} \left[ y + \frac{\beta}{2\lambda^{-1}} \right]^2 Y + \frac{\beta^2}{4\lambda^{-1}} Y = -E_s Y
\]

and introducing a new variable \( \eta = (\lambda^{-1})^{1/4} \left[ y + \frac{\beta}{2\lambda^{-1}} \right] \), we find the discrete energy:

\[
(E_y)_n = \sqrt{\lambda^{-1}} (2n + 1) - \frac{\beta^2}{4} (\lambda^{-1})^{3/2}.
\]

For Eq. (13), we have to use a special treatment, as the term \( |\psi_t|^2 \), which represents dynamics condensate, has several forms, depending on the available assumptions. For a better choice, by following the previous work of Pérez-García et al. [2] and using our space coordinates transformation, we chose the solution

\[
\psi_s = \frac{1}{\sqrt{\lambda} \pi} \frac{1}{2} \exp \left( -\frac{x^2}{2} \right) \exp(-i\omega t),
\]

where \( \rho \) denotes a dimensionless radial coordinate. To remove the \( \rho \) dependence in Eq. (13), we multiplied both sides with \( \exp(-\rho^2) \) and integrated over the space, thus yielding

\[
\frac{1}{Z} \frac{d^2 Z}{dz^2} - \frac{|\psi_s|^2}{2} \frac{3}{4} \sqrt{\lambda} \pi \cdot \frac{1}{2} \sqrt{\lambda} \pi \cdot \frac{1}{2} \exp(-z^2) = -E_z.
\]

Although Eq. (19) is literally a linear differential equation, an analytical solution is not available. However, because \( \alpha \) contains the parameter \( g_{sf} \), it represents the interaction between condensate and atom laser. In BEC, this parameter can be negative or positive, depending on the attractive (\( g_{sf} < 0 \)) or repulsive (\( g_{sf} > 0 \)) interaction between atoms in two states; i.e., \( s \) and \( \ell \). If we consider only the attractive interaction and enable approximating the exponent term in the second order,

\[
\exp(-z^2) \approx 1 - z^2,
\]

we get

\[
\frac{d^2 Z}{dz^2} - \frac{|\psi_s|^2}{2} \frac{3}{4} \sqrt{\lambda} \pi \cdot \frac{1}{2} \sqrt{\lambda} \pi \cdot \frac{1}{2} Z = -E_z + \frac{1}{Z} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} Z.
\]

Transforming \( \chi = \left[ |\psi_s|^2 \frac{1}{2} \lambda^2 \pi^2 \right]^{1/4} z \) and repeating the same procedure, we find the discrete energy

\[
(E_z)_r = \sqrt{\lambda^{-1}} (2r + 1).
\]

Inserting Eqs. (15), (17), and (22) into Eq. (14), we achieve

\[
(E_x)_{m} = \sqrt{\lambda^{-1}} (2m + 1).
\]
\[ E_{nm\sigma} = \left[ \sqrt{\lambda^{-1}} (2m + 2n) + \left| k \right| \frac{\sqrt{2}}{2} \frac{1}{4\pi} \frac{3}{2} (2r) \right] \]
\[ + 2\sqrt{\lambda^{-1}} + \left| k \right| \frac{\sqrt{2}}{2} \frac{1}{4\pi} \frac{3}{2} . \tag{23} \]

This result shows that the ground state of the system is indeed \( 2\sqrt{\lambda^{-1}} + \left| k \right| \frac{\sqrt{2}}{2} \frac{1}{4\pi} \frac{3}{2} \) and that degeneracy only occurs in the \( m \) and \( n \) states.

### 4. Conclusions

We have formulated the discrete energies of atom laser, which is weakly outcoupled from a condensate. This atom laser is naturally produced by pulsing the radio frequency to a condensate that includes two or more atoms, in order to flip the trapped state of the atoms to an untrapped state. In addition, because this extracted matter wave has a significant mass, it is accelerated by gravity, so that its de Broglie wavelength is naturally decreased.

It is also shown that the presence of anisotropic harmonic potential induces degeneracy only in two states, with the given eigenenergies stated in Eq. (23). Working in the Cartesian coordinates, these eigenenergies are obtained by resembling each of the differential equations to a harmonic quantum oscillator. For the \( z \) component, the discrete energy can only be gained by taking the attractive case between two or more atoms in two states, represented by \( g_{st} \). Furthermore, if we take the repulsive case, the solution to Eq. (21) does not result in discrete energy, as the left-hand side has an oscillation form.

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### References